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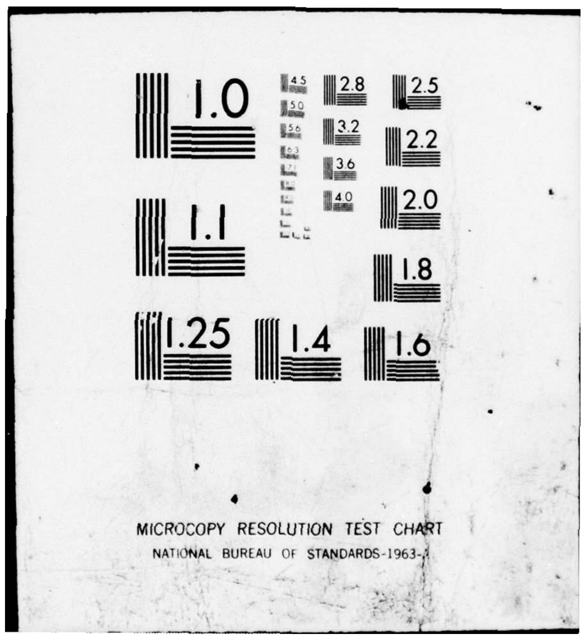
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A PROPERTY OF THE GAMMA DISTRIBUTION

KHURSHEED ALAM

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A PROPERTY OF THE GAMMA DISTRIBUTION

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ABSTRACT

Let U and V be independent random variables, and let W = UV. This paper concerns the distribution of U, given that V and W are distributed according to the gamma distributions. It is shown that U is distributed according to a beta distribution if the distributions of V and W are central gamma and that the distribution of U is degenerate at u = 1 if the distributions of V and W are non-central gamma. The given result is applied to determine the distribution of U when V and W are normally distributed.

Key words: Central and non-central gamma distributions.

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1. Main results. Let

$$g_m(x) = \frac{x^{m-1}}{\Gamma(m)} e^{-x}, \qquad x > 0$$

denote the gamma density function with m degrees of freedom. The non-central gamma distribiton $G_{m,\delta}$ with m degrees of freedom and non-centrality parameter equal to $\delta(>0)$ is given by the density function

$$(1.1) \qquad g_{m,\delta}(x) = e^{-\delta}\sum_{r=0}^{\infty} g_{m+r}(x)\,\frac{\delta^r}{r!}\,.$$

$$= e^{-\delta-x}\left(\frac{x}{\delta}\right)^{\frac{m-1}{2}} I_{m-1}(2\sqrt{x\delta})$$

where $I_m(x)$ denotes the modified Bessel function with parameter m.

Let U and V be independent random variables and let

$$(1.2) \qquad W = UV.$$

Suppose that the distribution of V is $G_{m,\delta}$ and the marginal distribution of W is $G_{m',\delta'}$. The main result of this paper concerns the distrubiton of U. Clearly U is positive with probability 1. Moreover

$$(1.3) \qquad P\{0 < U \le 1\} = 1.$$

Otherwise, let $P\{U \ge 1 + \xi\} = \alpha$, where ξ and α are positive numbers. Let $c = (1 + \xi)^{-1}$. Then

$$(1.4) \qquad Ee^{cW} = Ee^{cUV}$$

$$\ge \alpha E e^{V}$$

where E denotes expectation. The left hand side of (1.4) is finite, whereas the right hand side is infinite. Therefore (1.3) is true.

It is easy to show that $m' \leq m$ and $\delta' \leq \delta$. Let $E(U) = E(W)/E(V) = (m' + \delta')/(m + \delta) = \gamma$, say. The Laplace transform of the gamma distribution is given by

$$\int_0^\infty e^{-\lambda x}\, dG_{m,\delta}(x) = (1 + \lambda)^{-m} \exp\left(-\frac{\lambda\delta}{1 + \lambda}\right), \quad \lambda > -1.$$

Therefore, (1.2) yields

$$(1.5) \qquad (1 + \lambda)^{-m'} \exp\left(-\delta' + \frac{\delta'}{1 + \lambda}\right) = E(1 + \lambda U)^{-m} \exp\left(-\delta + \frac{\delta}{1 + \lambda U}\right)$$

$$\geq (1 + \lambda\gamma)^{-m} \exp\left(-\delta + \frac{\delta}{1 + \lambda\gamma}\right)$$

by Jensen's inequality. Comparing the two sides of (1.5) for large values of λ we find that $m' \leq m$.

Let

$$\Phi(a,b;x) = 1 + \frac{a}{b}x + \frac{a(a + 1)}{b(b + 1)} \frac{x^2}{2!} + \ldots$$

denote the confluent hypergeometric function. We have

$$EW^r = \frac{\Gamma(m' + r)}{\Gamma(m')} e^{-\delta'} \Phi(m' + r, m'; \delta'), \quad r > -m'.$$

Similarly

$$EV^r = \frac{\Gamma(m + r)}{\Gamma(m)} e^{-\delta} (m + r, m; \delta), \quad r > -m.$$

Therefore, (1.2) yields

$$(1.6)\qquad EU^r = \frac{\Gamma(m'+r)}{\Gamma(m')} e^{-\delta'} \Phi(m'+r, m'; \delta') \Big/ \frac{\Gamma(m+r)}{\Gamma(m)} e^{-\delta} \Phi(m+r, m; \delta)$$

$$= (\delta')^{\frac{3}{4} - \frac{m'}{2}} (\delta)^{\frac{m}{2} - \frac{3}{4}} r^{(m'-m)/2} \exp\left(\frac{\delta-\delta'}{2} - 2\sqrt{r}\right.$$

$$\left.(\sqrt{\delta} - \sqrt{\delta'})\right)(1 + O(r^{-1}))$$

for large values of r. The asymptotic expression given above is derived from Formula 6.13.2 (12) of Erdelyi (1953). Since $EU^r \le 1$ for $r \ge 0$, it follows from (1.6) that $\delta' \le \delta$.

Let H(u) denote the distribution function of U, and let $f_m^*(\nu) = (1+\nu)^{-m}$ and

$$f_m(\nu) = \int_0^1 (\nu + u)^{-m}\, dH(u), \qquad \nu > 0.$$

Putting $\nu = \frac{1}{\lambda}$ in (1.5) we get

$$(1.7)\qquad \frac{\nu^{m'-m}}{(1+\nu)^{m'}} \exp\left(-\frac{\delta'}{1+\nu}\right) = e^{-\delta} \int_0^1 (\nu+u)^{-m} \exp\left(\frac{\delta\nu}{\nu+u}\right) dH(u)$$

$$= e^{-\delta} \sum_{r=0}^{\infty} \frac{(-\delta\nu)^r\, \Gamma(m)}{r!\ \Gamma(m+r)} f_m^{(r)}(\nu)$$

$$= \Gamma(m)\, e^{-\delta} H_{m-1}(2\sqrt{\nu\delta D})\, f_m(\nu)$$

where $D = \frac{d}{d\nu}$ denotes the derivative operator with respect to ν, and

$$H_m(x) = \sum_{r=0}^{\infty} \frac{(-x^2/4)^r}{\Gamma(m+r+1)\, r!}.$$

Note that $J_m(x) = (\frac{x}{2})^m H_m(x)$ represents a Bessel function. Therefore

$$(1.8) \qquad (x \frac{d^2}{dx^2} + (2m - 1)\frac{d}{dx} + x)\ H_{m-1}(x) = 0.$$

Writing the left hand side of (1.7) in the same form as the right hand side we get

$$(1.9) \qquad \Gamma(m')\nu^{m'-m}\ e^{-\delta'}\ H_{m'-1}(2\sqrt{\theta\delta' D})f_{m'}{}^{*}(\nu) =$$

$$\Gamma(m)\ e^{-\delta}\ H_{m-1}\ (2\sqrt{\theta\delta D})\ f_m(\nu)$$

where $\theta = \nu$.

First let $m' = m$. A transformation of (1.8) gives

$$(1.10) \qquad (4x \frac{d^2}{dx^2} + 4m \frac{d}{dx} + c^2)\ H_{m-1}\ (c\sqrt{x}) = 0$$

where c is a constant. An application of the differential equation (1.10) to both sides of (1.9) with $c = 2\sqrt{\theta\delta}$ gives

$$4\theta(\delta - \delta')\ e^{-\delta'}\ H_{m-1}\ (2\sqrt{\theta\delta' D})\ f_m{}^{*}(\nu) = 0$$

or

$$(\delta - \delta')\ f_m{}^{*}(\nu)\ \exp\ (-\frac{\delta'}{1 + \nu}) = 0.$$

Therefore, $\delta = \delta'$. Hence, $P\{U = 1\} = 1$.

Next, let $m > m'$. If $\delta = 0$ then $\delta' = 0$, since $\delta' \leq \delta$, as shown above. Then (1.6) reduces to

$$EU^r = \frac{\Gamma(m' + r)}{\Gamma(m')} \Big/ \frac{\Gamma(m + r)}{\Gamma(m)}$$
$$= \int_0^1 u^r\ dH^*(u)$$

where H^* denotes the beta distribution $\beta(u;m',m-m')$. Hence, $H(u) = H^*(u)$. Suppose that $\delta > 0$. Writing

$$e^{-\delta'} \sum_{r=0}^{\infty} \binom{m-m'+r-1}{r} \Gamma(m+r)\, H_{m+r-1}\, (2\sqrt{\theta\delta' D})\, f^*_{m+r}(\nu)$$

for the left hand side of (1.7) and applying the differential equation (1.10) to both sides with $c = 2\sqrt{\theta\delta}$ we get

$$e^{-\delta'} \sum_{r=0}^{\infty} \binom{m-m'+r-1}{r} \Gamma(m+r)\left(4\theta(\delta-\delta')-4r\,\frac{d}{dD}\right)$$

$$H_{m+r-1}(2\sqrt{\theta\delta' D})\, f^*_{m+r}(\nu) = 0$$

or

$$4\theta(\delta-\delta')\,\frac{\nu^{m'-m}}{(1+\nu)^{m'}} \exp\left(-\frac{\delta'}{1+\nu}\right) + 4\theta\delta' \sum_{r=0}^{\infty} r\binom{m-m'+r-1}{r}$$

$$\sum_{s=0}^{\infty} \frac{(\theta\delta')^s}{s!}\,\frac{(1+\nu)^{m-r-s}}{m+r+s} = 0.$$

The above equation implies that $\delta = \delta' = 0$, contrary to the assumption that $\delta > 0$.

The foregoing results are summarized in the following theorem.

<u>Theorem 1</u>. Let $W = UV$ where U and V are independent random variables, and let V and W be distributed according to the gamma distributions $G_{m,\delta}$ and $G_{m',\delta'}$, respectively. Then $m' \leq m$, $\delta' \leq \delta$ and $P\{0 < U \leq 1\} = 1$. If $m' = m$ then $\delta' = \delta$ and $P\{U = 1\} = 1$. If $m' < m$ then $\delta' = \delta = 0$ and U is distributed according to the beta distribution $\beta(u;m',m-m')$.

Since the square of a normal random variable is distributed according to the gamma distribution with a scale factor we obtain the following corollary from the above theorem.

<u>Corollary 1</u>. Let $W = UV$ where U and V are independent random variables, and let V and W be normally distributed with unit variance and means equal to μ and μ', respectively. Then $\mu^2 = \mu'^2$ and $P\{U^2 = 1\} = 1$. If $\mu = -(+)\mu' \neq 0$ then $P\{U = -(+)1\} = 1$.

An extension of Corollary 1 is given as follows: Let $\alpha_1, \ldots, \alpha_p$, $Z_1, \ldots, Z_p$ be independent random variables and let $Z = \sum_{i=1}^{p} \alpha_i Z_i$. Let $EZ_i = \xi_i$.

<u>Corollary 2</u>. If the random variables $Z, Z_1, \ldots, Z_p$ are normally distributed then $P\{\alpha_i = c_i\} = 1$ when $\xi_i \neq 0$ and $P\{\alpha_i^2 = c_i^2\} = 1$ when $\xi_i = 0$ for each $i = 1, \ldots, p$ where $c_1, \ldots, c_p$ are certain constants.

<u>Proof</u>. Since Z is normally distributed it follows from the reproductive property of the normal distribution (see e.g., Lukacs and Laha (1964) Lemma 5.1.1) that $\alpha_i Z_i$ is normally distributed for each $i = 1, \ldots, p$. The conclusion of the corollary follows from Corollary 1.

Let $\alpha_1, \ldots, \alpha_p$ be m-component random vectors and let A denote the matrix whose ith column vector is α_i, $i = 1, \ldots, p$. Let $Z_1, \ldots, Z_p$ be p independent normal random variables. Let

$EZ_i = \xi_i$, $Z = (Z_1,\ldots,Z_p)'$ and $Y = AZ$. The random vector Y is distributed according to a multivariate normal distribution if and only if $\lambda' Y = \sum_{i=1}^{p} (\lambda'\alpha_i)Z_i$ is normally distributed for every non-null vector λ. The following result follows from Corollary 2.

Corollary 3. Let $\alpha_1,\ldots,\alpha_p$, Z be independent. If the distribution of Y is multivariate then $P\{\lambda'\alpha_i = c_i\} = 1$ when $\xi_i \neq 0$ and $P\{(\lambda'\alpha_i)^2 = c_i^2\}$ when $\xi_i = 0$, $i = 1,\ldots,p$ for each non-null vector λ, where $c_1,\ldots,c_p$ are certain constants depending on λ.

Corollary 3 is related to the following result due to Kingman and Graybill (1970). Let $Y_1,\ldots,Y_p$ be independent and identically distributed random variables and let $A = (a_{ij})$ be a $p\times p$ random matrix which is orthogonal with probability 1 and $E(\sum_{j=1}^{p} a_{ij}) \neq 0$ for some i. Let $Y = (Y_1,\ldots,Y_p)'$ and $Z = AY$. Then the components of Z are independently and identically distributed according to the standard normal distribution if and only if the components of Y have the same distribution.

Theorem 2 below gives a characterization of the gamma distribution. Let

$$F(a,b;c;x) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\ \Gamma(b+r)\ \Gamma(c)}{\Gamma(a)\ \Gamma(b)\ \Gamma(c+r)} \frac{x^r}{r!}$$

denote the hypergeometric function, and let

$$\phi(\lambda) = F(a,b;c;-\lambda) \tag{1.11}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\ \Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1 + t\lambda)^{-a}dt$$

$$a,b,c > 0,\ c > b.$$

It is seen that $\phi(0) = 1$ and that $\phi(\lambda)$ is a completely monotone function, that is, $(-1)^r \phi^{(r)}(\lambda) \geq 0$, $\lambda > 0$. Therefore, $\phi(\lambda)$ represents the Laplace transform of a probability distribution on $[0,\infty)$. A distribution on $[0,\infty)$, whose Laplace transform is given by (1.11) will be called inverse-hypergeometric. If $b = c$ then $\phi(\lambda) = (1 + \lambda)^{-a}$ is the Laplace transform of the gamma distribution with a degrees of freedom.

Theorem 2. Let $W = UV$ where U and V are independent random variables, and let U be distributed according to a beta distribution $\beta(u;p,q)$, say. Then W is distributed according to a gamma distribution with m degrees of freedom if and only if $m \leq p$ and the distribution V is inverse-hypergeometric, given by the Laplace transform $\phi(\lambda) = F(p+2;m;p;-\lambda)$.

Proof. Suppose that the Laplace transform of the distribution of V is given by $\phi(\lambda) = F(p+q;m;p;-\lambda)$. The rth moment of V is given by

$$EV^r = \frac{\Gamma(p+q+r)\ \Gamma(m+r)\ \Gamma(p)}{\Gamma(p+q)\ \Gamma(m)\ \Gamma(p+r)} . \tag{1.12}$$

Therefore,

$$\begin{aligned} EW^r &= EU^r EV^r \\ &= \frac{\Gamma(m+r)}{\Gamma(m)} . \end{aligned} \tag{1.13}$$

The right hand side of (1.13) represents the rth moment of the gamma distribution with m degrees of freedom. Therefore, the distribution of W is gamma.

Next, suppose that the distribution of W is gamma with m degrees of freedom. Considering the Laplace transform of the distribution of W, we have

$$(1.14)\qquad (1+\lambda)^{-m} = Ee^{-\lambda W}$$
$$= Ee^{-\lambda UV}$$
$$= E\,\frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\int_0^1 u^{p-1}(1-u)^{q-1}e^{-uV}du$$
$$= E\,\Phi(p,p+q;-\lambda V)$$
$$\geq \Phi(p,p+q;-\lambda EV)$$
$$= \frac{\Gamma(p+q)}{\Gamma(q)}\,\lambda^{-p}(1+0(\lambda^{-1}))\ \text{as}\ \lambda\to\infty.$$

It follows from (1.14) that $m \leq p$.

The rth moment of V is given by

$$EV^r = EW^r/EU^r$$
$$= \frac{\Gamma(m+r)}{\Gamma(m)}\cdot\frac{\Gamma(p+q+r)\,\Gamma(p)}{\Gamma(p+q)\,\Gamma(p+r)}$$

From (1.12) it follows that the Laplace transform of the distribution of V is given by $\phi(\lambda) = F(p+q;m;p;-\lambda)$. Hence the distribution of V is inverse-hypergeometric.

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Let U and V be independent random variables, and let W = UV. It is shown that if the distribution of V and W are central gamma then the distribution of V is beta, and if the distributions of V and W are non-central gamma then the distribution of U is degenerate at u = 1.

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